Journal of Statistical Physics, Vol. 38, Nos. 3/4, 1985

Some Properties of Random Ising Models

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Received May 30, 1984; revised July 25, 1984

We consider an Ising model with random magnetic field h_i and random nearest-neighbor couplings J_{ij}. The random variables h_i and J_{ij} are independent and identically distributed with a nice enough distribution, e.g., Gaussian. We will prove that (i) at high temperature, infinite volume correlation functions are independent on the boundary conditions and decay exponentially fast with probability 1 and (ii) for any temperature with sufficiently strong magnetic field the correlation functions are again independent on the boundary conditions and decay exponentially fast with probability 1. We also prove that the averaged magnetization of the ground state configuration of the one-dimensional Ising model with random magnetic field is zero, no matter how small is the variance of the h_i.

KEY WORDS: Spin systems; cluster expansion; spin glasses; random field Ising models.

1. DEFINITION OF THE MODEL AND STATEMENT OF THE RESULTS

We consider an Ising model defined by the following Hamiltonian:

$$H = -\sum_{(ij)\in\bar{\Lambda}^*} J_{ij}\sigma_i\sigma_j - \varepsilon\sum_{i\in\Lambda} h_i\sigma_i \tag{1}$$

Here, $\Lambda \subset \mathbb{Z}^d$ and $\bar{\Lambda}$ is the union of Λ and the set $\partial\Lambda$ of its nearest neighbors; the values of the spins on $\partial\Lambda$ are fixed and determine the boundary conditions. If X is a set of sites, X^* denotes the set of its nearest-neighbor bonds.

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0022-4715/85/0200-0483$04.50/0

If $F(\sigma)$ is a function of the spins, its thermal expectation is defined by

$$\langle F(\sigma)\rangle_{\beta,\Lambda} = \frac{1}{Z_{\beta,\Lambda}} \langle F(\sigma)\, e^{-\beta H_\Lambda}\rangle_0 \tag{2}$$

where $\langle\cdot\rangle_0$ is the normalized Ising product measure:

$$\langle\cdot\rangle_0 = \frac{1}{2^{|\Lambda|}} \sum_{(\sigma_i=\pm 1)_{i\Lambda}} \cdot \tag{3}$$

and $Z_{\beta,\Lambda}$ is the partition function.

The couplings $\{J_{ij}\}$ are independent and identically distributed random variables whose distribution $d\mu(J)$ satisfies the following condition:

$$\int d\mu(J)\, e^{a|J|} < \infty \qquad \forall a > 0 \tag{4}$$

The magnetic field variables $\{h_i\}$ are independent and identically distributed random variables with a symmetric distribution such that $P[h=0]=0$, i.e., such that there is no "mass" at zero.

Disorder averages (i.e., expectations with respect to the J_{ij}s and the h_is) are denoted by $\mathbb{E}[\cdot]$. The supremum with respect to the spin variables will be denoted by $\|\cdot\|_\infty$.

We will prove the following results.

Theorem 1. Let $\beta < \beta_0$, where β_0 is a constant depending only on the parameters of the model. Then the following cluster property holds uniformly in Λ:

$$\mathbb{E}[|\langle F; G\rangle_{\beta,\Lambda}|] = \mathbb{E}[|\langle FG\rangle_{\beta,\Lambda} - \langle F\rangle_{\beta,\Lambda}\langle G\rangle_{\beta,\Lambda}|] \leqslant c e^{-m d_{F,G}}$$

where

$$d_{F,G} = \min_{\substack{i\in \mathrm{suppt}\, F\\ j\in \mathrm{suppt}\, G}} \|i-j\|$$

is the distance between the supports of F and G.

Moreover, the following infinite volume limit exists and is independent on the boundary conditions:

$$\langle\langle F\rangle\rangle_\beta = \lim_{\Lambda\to Z^d} \mathbb{E}[\langle F\rangle_{\beta,\Lambda}]$$

while thermal expectations $\langle\cdot\rangle_{\beta,\Lambda}$ satisfy the cluster property and have an infinite volume limit independent of the boundary conditions with probability 1.

Theorem 2. Let $\varepsilon > \varepsilon_0$, where ε_0 is a constant depending only on the parameters of the model. Then the same results proved in the high-temperature case hold. Moreover, the infinite volume averaged magnetization vanishes for any choice of the boundary conditions.

Clearly Theorem 1 is trivial if the couplings are bounded random variables. In this case in fact we could prove converge of the cluster expansion uniformly in the couplings.

The technique used to prove Theorems 1 and 2 is a cluster expansion of the type originally developed in Ref. 1. [For a different approach to cluster expansions, see Seiler (Ref. 1).] The fact that we are considering a lattice spin system instead of a continuum quantum field simplifies most of the expansion.

In addition, we will prove the following result for the one-dimensional Ising model with random magnetic field.

Theorem 3. Let

$$H_L = -J \sum_{i=-L-1}^{L} \sigma_i \sigma_{i+1} - \varepsilon \sum_{i=-L}^{L} h_i \sigma_i$$

$$\sigma_{-L-1} = \sigma_{L+1} = +1, \qquad \sigma_i = \pm 1 \text{ for } i = -L \cdots L$$

Let the h_i be i.i.d. Gaussian random variables of mean zero and variance 1, and for each realization $\{h_j\}$ of the random magnetic field let $\{\sigma_i^*(h)\}$ be the ground state (i.e., zero temperature) configuration of H_L. Then for each $\varepsilon > 0$ we have

$$\mathbb{E}[\sigma_i^*] \to 0 \qquad \text{as} \quad L \to \infty$$

and

$$\mathbb{E}[\sigma_i^* \sigma_j^*] \leqslant \bar{c} e^{-\bar{m}|i-j|} \text{ uniformly in } L$$

2. THE CLUSTER EXPANSION: HIGH-TEMPERATURE CASE

Theorem 1 follows from a standard high-temperature expansion. Here the problem is that, the couplings being unbounded random variables, there are regions where they are so large that to expand there seems hopeless. The point is that this happens with a small probability, so upon averaging our expansion converges.

To implement this idea, we introduce the interpolating parameters $\{s_b\}_{b \in \bar{\Lambda}^*}$, $s_b \in [0, 1]$ in the Hamiltonian:

$$H_\Lambda(s) = - \sum_{(ij) \in \bar{\Lambda}^*} s_{ij} J_{ij} \sigma_i \sigma_j - \varepsilon \sum_{i \in \Lambda} h_i \sigma_i$$

We define, for $\Gamma \subset (\mathbb{Z}^d)^*$,

$$(s^\Gamma)_b = s_b \qquad \text{if} \quad b \in \Gamma$$
$$= 0 \qquad \text{if} \quad b \notin \Gamma$$

In other words, $H_\Lambda(s^\Gamma)$ has "zero boundary conditions" (i.e., no coupling) on $\Gamma^c = (\mathbb{Z}^d)^* \backslash \Gamma$.

Let moreover

$$\partial_b = \frac{\partial}{\partial s_b}, \qquad \partial_\Gamma = \prod_{b \in \Gamma} \partial_b, \qquad \int ds^\Gamma = \prod_{b \in \Gamma} \int_0^1 ds'_b$$

If $X \subset \Lambda$, $(X)_\Gamma$ will denote the family of connected components of X obtained by deleting from X^* all the bonds but the ones in Γ.

By a standard technique (see Ref. 1) we then obtain the following expansion:

$$\langle F \rangle_{\beta,\Lambda} = \sum_{X,\Gamma}{}^* \frac{Z_{\Lambda \backslash X}}{Z_\Lambda} \int ds^\Gamma \, \partial_\Gamma \langle F e^{-\beta H_X(s^\Gamma)} \rangle_0$$

where $\sum^*_{X,\Gamma}$ means sum over $X \subset \Lambda$, $\Gamma \subset X^*$ such that X is connected and each connected component of $(X)_\Gamma$ has nonempty intersection with the support of F.

We have the following proposition.

Proposition 1. Under the hypotheses of Theorem 1 the following cluster expansion:

$$\mathbb{E}[|\langle F \rangle_{\beta,\Lambda}|] = \sum_{X,\Gamma}{}^* \mathbb{E}\left[\frac{Z_{\Lambda \backslash X}}{Z_\Lambda} \left| \int ds^\Gamma \, \partial_\Gamma \langle F e^{-\beta H_X(s^\Gamma)} \rangle_0 \right| \right]$$

converges uniformly in Λ exponentially fast, that is,

$$\sum_{\substack{X,\Gamma \\ |X| \geqslant n}}{}^* \mathbb{E}\left[\frac{Z_{\Lambda \backslash X}}{Z_\Lambda} \left| \int ds^\Gamma \, \partial_\Gamma \langle F e^{-\beta H_X(s^\Gamma)} \rangle_0 \right| \right] \leqslant K_1 e^{-K_2 n}$$

where K_1 is independent on Λ and K_2 is independent on F and Λ.

The proof is a straightforward consequence of Lemmas 1, 2, and 3 below.

Theorem 1 is proved along the same lines of Proposition 1, the only difference being that we need to use duplicate variables to obtain the bound for the truncated correlation functions. It is therefore useful to outline its proof.

Proof of Theorem 1. Consider two independent, identical copies of the same system defined by (1), (2), (3); we denote by $\sigma^{(1)}$ (resp. $\sigma^{(2)}$) the spins in the first (resp. second) system. We define "duplicated" expectations as follows:

$$\langle \cdot \rangle_{\beta,\Lambda}^{(\text{dup})} = \frac{\sum_{(\sigma_i^{(1)} = \pm 1, \sigma_j^{(2)} = \pm 1)_{i,j \in \Lambda}} e^{-\beta H_\Lambda^{(1)} - \beta H_\Lambda^{(2)}}}{Z_\Lambda^{(\text{dup})}} \tag{5}$$

where $H_\Lambda^{(\alpha)} = H_\Lambda(\sigma^{(\alpha)})$, $\alpha = 1, 2$ and clearly $Z_\Lambda^{(\text{dup})} = Z_\Lambda^2$.

The duplicated expectation so defined is clearly symmetric with respect to the symmetry $\sim$ which sends variables of each system into variables of the other.

If F is a function of the spins, we denote by $F^{(1)}, F^{(2)}$ its representatives in each subsystem.

We have

$$2\langle F; G \rangle_{\beta,\Lambda} = \langle (F^{(1)} - F^{(2)})(G^{(1)} - G^{(2)}) \rangle_{\beta,\Lambda}^{(\text{dup})}$$

both terms inside the brackets are odd under $\sim$ and their product is even. Now, we apply the cluster expansion to (5):

$$2\mathbb{E}[|\langle F; G \rangle_{\beta,\Lambda}|]$$
$$= 2\mathbb{E}\left[\left|\sum_{X,\Gamma}{}^{*} \frac{Z_{\Lambda\setminus X}^2}{Z_\Lambda^2} \int ds^\Gamma \, \partial_\Gamma \langle F'G' e^{-\beta H_X^{(1)}(s^\Gamma) - \beta H_X^{(2)}(s^\Gamma)} \rangle_0^{(\text{dup})}\right|\right] \tag{6}$$

where $F' = F^{(1)} - F^{(2)}$, $G' = G^{(1)} - G^{(2)}$. Now each term in the cluster expansion is even under $\sim$; moreover, since the Hamiltonians $H_X^{(1)}(s^\Gamma)$, $H_X^{(2)}(s^\Gamma)$ do not couple through Γ^c, each term in the cluster expansion is invariant under a $\sim$ transformation applied separately on each connected component of (X). Since F', G' are odd, it follows that each nonzero term in the expansion must have connected components each connecting both $\operatorname{suppt} F$ and $\operatorname{suppt} G$. Hence, for each nonzero term in (6) it must be $|X| \geqslant (\text{const})\, d_{F,G}$.

Next, we show how to bound (6). We have

$$(6) \leqslant 2 \sum_{X,\Gamma}{}^{*} \mathbb{E}\left[\frac{Z_{\Lambda\setminus X}^2}{Z_\Lambda^2} \left|\int ds^\Gamma \, \partial_\Gamma \langle F'G' e^{-\beta H_X^{(1)}(s^\Gamma) - \beta H_X^{(2)}(s^\Gamma)} \rangle_0^{(\text{dup})}\right|\right]$$

We apply Schwartz' inequality and by Lemmas 1, 2, and easy variant of Lemma 3, together with the fact that $|X| \geqslant (\text{const})\, d_{F,G}$, we prove that

$$\mathbb{E}[|\langle F; G \rangle_{\beta,\Lambda}|] \leqslant c' e^{-m d_{F,G}} \tag{7}$$

The statement with probability 1 follows from an easy argument. If $\mathscr{T}$ is a set of nonzero measure in J space where the cluster property does not hold then

$$\mathbb{E}[ce^{-m(J)d}] \geqslant c''e^{-md}\mathbb{P}[\mathscr{T}^c] + c'''e^{-m'd}\mathbb{P}[\mathscr{T}]$$

with $m' < m$, so that (7) does not have a chance to hold. It follows that $\mathbb{P}[\mathscr{T}] = 0$.

The existence of an infinite volume limit independent of the boundary conditions is now trivial to prove. This completes the proof. ■

We now present the three main technical estimates used in the above proof.

Lemma 1.

$$\sum_{\substack{X,\Gamma \\ |X|=n}}^{*} 1 = k_3 e^{k_4 n}$$

Lemma 2.

$$\mathbb{E}\left[\left(\frac{Z_{\Lambda\setminus X}}{Z_\Lambda}\right)^r\right]^{1/r} \leqslant e^{k_5|X|}$$

Lemma 3.

$$\mathbb{E}\left[\left(\int ds^\Gamma\, \partial_\Gamma \langle F e^{-\beta H_X(s^\Gamma)}\rangle_0\right)^r\right]^{1/r} \leqslant K_6 e^{K_7|X| - K_8|\Gamma|}$$

where $K_8 \to \infty$ as $\beta \to 0$.

The r powers appear because we have to apply the Schwartz inequalities to separate the various J-dependent terms in the expansion.

The proof of Lemma 1 is standard (see Ref. 1, first reference, Prop. 5.1) and Lemma 3 will be proven in Section 5, so we prove now only Lemma 2.

Proof of Lemma 2. This is a simple stability estimate, made even simpler by the boundedness of the spins. In fact we have

$$\frac{Z_{\Lambda\setminus X}}{Z_\Lambda} = \left\langle \exp\left(\beta \sum_{(ij)\in \bar{X}^*} J_{ij}\sigma_i\sigma_j\right)\right\rangle_{\beta,\Lambda}$$

$$\leqslant \left\langle \exp\left(\beta \sum_{(ij)\in \bar{X}^*} |J_{ij}|\right)\right\rangle_{\beta,\Lambda} \leqslant e^{O(|X|)}$$

Lemma 2 then follows by condition (4). This completes the proof. ■

3. THE CLUSTER EXPANSION: LARGE FIELD CASE

The problem in applying the cluster expansion is that the inverse temperature β is no longer small and we must find convergence factors elsewhere. A simple calculation shows that, if the field is large, the magnetization should follow the field. In fact, by summing over $\sigma_j = \pm 1$, $j \neq i$, we find

$$\langle \sigma_i \rangle_{\beta,\Lambda} = \frac{\langle \sinh(\beta \sum_{l:\|l-i\|=1} J_{li}\sigma_l + \beta\varepsilon h_i)\rangle_{\beta,\Lambda\setminus\{i\}}}{\langle \cosh(\beta \sum_{l:\|l-i\|=1} J_{li}\sigma_l + \beta\varepsilon h_i)\rangle_{\beta,\Lambda\setminus\{i\}}}$$

and thus $\langle \sigma_i \rangle_{\beta,\Lambda}$ has the same sign as h_i with a probability near 1 by choosing ε large enough.

We remark also that $\mathbb{E}[\langle \sigma_i \rangle_{\beta,\Lambda}]$ is clearly zero for all β, ε, Λ if we have free boundary conditions, by symmetry. Hence, the second part of Theorem 2 is simply a corollary of the first part: if we can show that $\mathbb{E}[\langle \sigma_i \rangle_{\beta,\Lambda}]$ has a unique infinite volume limit independent on the boundary conditions, then this limit has to be equal to the one obtained by taking free boundary conditions, which is zero by the above argument.

Since heuristically the magnetization "follows the field," it is natural to consider the following change of variables:

$$\psi_i = \sigma_i - \hat{h}_i$$

where

$$\begin{aligned} \hat{h}_i &= \operatorname{sign}(h_i) \qquad \text{if} \quad |h_i| > B \\ &= 0 \qquad \text{if} \quad |h_i| \leqslant B \end{aligned}$$

ψ_i should be "small" if the field is large, and therefore a cluster expansion in the ψ variables should converge.

We introduce a new normalized product measure:

$$\langle f(\psi) \rangle_0^{\psi} = z(\beta, \varepsilon, h)^{-1} \sum_{(\psi_i = \pm 1 - \hat{h}_i)_{i \in \Lambda}} \left(\prod_{i \in \Lambda} e^{\beta \varepsilon h_i \psi_i} \right) f(\psi)$$

where $z(\beta, \varepsilon, h)$ is a normalization.

Next, for $G(\psi)$ a function of the ψ variables we define

$$\langle G(\psi) \rangle_{\beta,\Lambda}^{\psi} = \frac{1}{Z_{\beta,\Lambda}^{\psi}} \langle G(\psi)\, e^{-\beta H_\Lambda^{\psi}} \rangle_0^{\psi}$$

$$H_\Lambda^{\psi} = -\sum_{(ij) \in \bar{\Lambda}^*} J_{ij}(\psi_i \psi_j + \hat{h}_i \psi_j + \hat{h}_j \psi_i) - \varepsilon \sum_{i \in \Lambda} h_i \psi_i$$

with the obvious meaning of $Z^{\psi}_{\beta,\Lambda}$. We dropped from the Hamiltonian H^{ψ}_{Λ} terms which are "constant" (i.e., independent on ψ), since they would cancel upon normalization.

Clearly, these expectations have been defined in such a way that

$$\langle F(\psi+\hat{h})\rangle^{\psi}_{\beta,\Lambda} = \langle F(\sigma)\rangle_{\beta,\Lambda}$$

We introduce, as usual, interpolating parameters in the Hamiltonian:

$$H^{\psi}_{\Lambda}(s) = -\sum_{(ij)\in\bar{\Lambda}^*} J_{ij}(s_{ij}\psi_i\psi_j + \hat{h}_i\psi_j + \hat{h}_j\psi_i) - \varepsilon \sum_{i\in\Lambda} h_i\psi_i$$

Again, we have a cluster expansion in the ψ variables like the one in the preceding section:

$$\langle F(\psi+\hat{h})\rangle^{\psi}_{\beta,\Lambda} = \sum_{X,\Gamma}{}^{*} \frac{Z^{\psi}_{\Lambda\backslash X}}{Z^{\psi}_{\Lambda}} \int ds^{\Gamma}\, \partial_{\Gamma}\langle F(\psi+\hat{h})\, e^{-\beta H^{\psi}_{X}(s^{\Gamma})}\rangle^{\psi}_{0}$$

As in the high-temperature case, the convergence of the cluster expansion for $\mathbb{E}[\langle F\rangle_{\beta,\Lambda}]$ and Theorem 2 follow from Lemmas 1, 4, and 5 below.

Lemma 4:

$$\mathbb{E}\left[\left(\frac{Z^{\psi}_{\Lambda\backslash X}}{Z^{\psi}_{\Lambda}}\right)^{r}\right]^{1/r} \leqslant e^{K_9|X|}$$

Lemma 5:

$$\mathbb{E}\left[\left(\int ds^{\Gamma}\, \partial_{\Gamma}\langle F(\psi+\hat{h})\, e^{-\beta H^{\psi}_{X}(s^{\Gamma})}\rangle^{\psi}_{0}\right)^{r}\right]^{1/r} \leqslant K_{10} e^{K_{11}|X| - K_{12}|\Gamma|}$$

where $K_{12}\to\infty$ as $\varepsilon\to\infty$.

The proof of Lemma 4 is almost identical to the proof of Lemma 2. Lemma 5 will be proved in the following section.

4. THE MAIN ESTIMATES

In this section we finish the proof of the convergence of the cluster expansion developed in Sections 2 and 3 by the proving Lemmas 3 and 5.

Proof of Lemma 3. We compute the s derivatives and bound the result:

$$|\partial_{\Gamma}\langle F e^{-\beta H_X(s^{\Gamma})}\rangle_0| = \left|\left\langle F e^{-\beta H_X(s^{\Gamma})} \left(\prod_{(hk)\in\Gamma} \beta J_{hk}\sigma_h\sigma_k\right)\right\rangle_0\right|$$

$$\leqslant \beta^{|\Gamma|}\cdot \left|\prod_{(hk)\in\Gamma} J_{hk}\right| \cdot \|F\|_{\infty} \langle e^{-\beta H_X(s^{\Gamma})}\rangle_0$$

For some values s^* of the interpolating parameters $H_X(s^\Gamma)$ has a minimum, since it is a continuous function over the compact set $[0, 1]^{|A^*|}$; it follows that we can take the supremum over the s and bound the s integrals in the obvious way. Next we square and take the disorder expectation, to obtain the bound:

$$\mathbb{E}\left[\left(\int ds^\Gamma\, \partial_\Gamma \langle P e^{-\beta H_X(s^\Gamma)}\rangle_0\right)^2\right] \leqslant \beta^{2|\Gamma|}\, \|F\|_\infty^2\, \mathbb{E}\left[\left(\prod_{(hk)\in\Gamma} J_{hk}\right)^4\right]^{1/2} \mathbb{E}\,[\langle e^{-\beta H_X(s^*)}\rangle_0^4]^{1/2}$$

Then we have

$$\mathbb{E}\left[\prod_{(hk)\in\Gamma} J_{hk}^4\right]^{1/2} = \prod_{(hk)\in\Gamma} \mathbb{E}[J_{hk}^4]^{1/2} = e^{K_{13}|\Gamma|}$$

and by a stability estimate whose proof is analogous to the one of Lemma 2:

$$\mathbb{E}[\langle e^{-\beta H_X(s^*)}\rangle_0^4]^{1/2} \leqslant e^{K_{14}|X|}$$

Summarizing, we have obtained the following bound:

$$\mathbb{E}\left[\left(\int ds^\Gamma\, \partial_\Gamma \langle F e^{-\beta H_X(s^\Gamma)}\rangle_0\right)^2\right]^{1/2} \leqslant \beta^{|\Gamma|}\, \|F\|_\infty\, e^{(1/2)K_{13}|\Gamma| + (1/2)K_{14}|X|}$$

Now, take $K_6 = \|F\|_\infty$, $K_7 = (1/2)\,K_{14}$, and $K_8 = \log(1/\beta) - (1/2)\,K_{13}$ and for β small enough we have Lemma 3 in the case $r = 2$. The general $r \geqslant 2$ case follows then easily.

The proof is complete. ■

Proof of Lemma 5. The main difference with the proof of Lemma 3 is that we must use the fact that ψ is in some sense small, rather than β.

We start computing the s derivatives as before:

$$\begin{aligned}
&|\partial_\Gamma \langle F(\psi + \hat{h})\, e^{-\beta H_X^\psi(s^\Gamma)}\rangle_0^\psi| \\
&\quad = \left|\left\langle F(\psi + \hat{h})\, e^{-\beta H_X^\psi(s^\Gamma)} \left(\prod_{(hk)\in\Gamma} \beta J_{hk} \psi_h \psi_k\right)\right\rangle_0^\psi\right| \\
&\quad \leqslant \beta^{|\Gamma|} \cdot \|F(\psi + \hat{h})\|_\infty \cdot \left|\prod_{(hk)\in\Gamma} J_{hk}\right| \cdot \left|\left\langle e^{-\beta H_X^\psi(s^\Gamma)} \left(\prod_{(hk)\in\Gamma} \psi_h \psi_k\right)\right\rangle_0^\psi\right| \\
&\quad \leqslant \beta^{|\Gamma|} \cdot \|F(\psi + \hat{h})\|_\infty \cdot \left|\prod_{(hk)\in\Gamma} J_{hk}\right| \cdot \|e^{-\beta H_X^\psi(s^*)}\|_\infty \\
&\qquad \cdot \left\langle \left|\prod_{(hk)\in\Gamma} \psi_h \psi_k\right|\right\rangle_0^\psi,
\end{aligned}$$

where as before we minimized H_X^ψ over the interpolating parameters.

We take the disorder expectation of the above squared and we obtain

$$\mathbb{E}\left[\left(\prod_{(hk)\in\Gamma} J_{hk}^2\right)\cdot \|e^{-\beta H_X(s^*)}\|_\infty^2 \left(\left\langle\left|\prod_{(hk)\in\Gamma}\psi_h\psi_k\right|\right\rangle_0^\psi\right)^2\right]$$
$$\cdot\,\beta^{2|\Gamma|}\,\|F(\psi+\hat{h})\|_\infty^2$$

Next, we apply a Hölder inequality to obtain

$$\beta^{2|\Gamma|}\,\|F(\psi+\hat{h})\|_\infty^2\,\mathbb{E}\left[\prod_{(hk)\in\Gamma} J_{hk}^6\right]^{1/3}\cdot\mathbb{E}\left[\|e^{-\beta H_X^\psi(s^*)}\|_\infty^6\right]^{1/3}$$
$$\cdot\,\mathbb{E}\left[\left(\left\langle\left|\prod_{(hk)\in\Gamma}\psi_h\psi_k\right|\right\rangle_0^\psi\right)^6\right]^{1/3} \tag{8}$$

The first two $\mathbb{E}[\cdot]$ expectations are clearly bounded as follows:

$$\mathbb{E}\left[\prod_{(hk)\in\Gamma} J_{hk}^6\right]^{1/3}\leqslant e^{K_{15}|\Gamma|}$$
$$\mathbb{E}[\|e^{-\beta H_X^\psi(s^*)}\|_\infty^6]^{1/3}\leqslant e^{K_{16}|X|}$$

The third factor deserves more attention. First of all we apply a Jensen inequality and a Hölder inequality to obtain

$$\mathbb{E}\left[\left(\left\langle\left|\prod_{(hk)\in\Gamma}\psi_h\psi_k\right|\right\rangle_0^\psi\right)^6\right]\leqslant\mathbb{E}\left[\left\langle\prod_{(hk)\in\Gamma}(\psi_h\psi^k)^6\right\rangle_0^\psi\right]$$
$$\leqslant\left(\prod_{(hk)\in\Gamma}\mathbb{E}\langle\psi_h^{12d}\rangle_0^\psi\,\mathbb{E}[\langle\psi_k^{12d}\rangle_0^\psi]\right)^{1/2d}$$

Next we compute explicitly $\langle\psi^{12d}\rangle_0^\psi$:

$$\langle\psi^{12d}\rangle_0^\psi=\frac{e^{-\beta\varepsilon h(\hat{h}-1)}(\hat{h}-1)^{12d}+e^{-\beta\varepsilon h(\hat{h}+1)}(\hat{h}+1)^{12d}}{e^{-\beta\varepsilon h\hat{h}}(e^{\beta\varepsilon h}+e^{-\beta\varepsilon h})}$$
$$=\frac{e^{\beta\varepsilon h}(\hat{h}-1)^{12d}+e^{-\beta\varepsilon h}(\hat{h}+1)^{12d}}{e^{\beta\varepsilon h}+e^{-\beta\varepsilon h}}$$
$$=\begin{cases}1, & \text{if}\quad |h|\leqslant B\\ 2^{12d}\,\dfrac{e^{-2\beta\varepsilon|h|}}{1+e^{-2\beta\varepsilon|h|}}, & \text{if}\quad |h|>B\end{cases}$$

It follows that

$$\mathbb{E}[\langle\psi^{12d}\rangle_0^\psi]=\mathbb{P}[|h|\leqslant B]+\mathbb{P}[|h|>B]\,\mathbb{E}\left[\frac{2^{12d}e^{-2\beta\varepsilon|h|}}{1+e^{-2\beta\varepsilon|h|}}\,\middle|\,|h|>B\right]$$
$$\leqslant\mathbb{P}[|h|\leqslant B]+2^{12d}e^{-2\beta\varepsilon B}$$

Now, choose B to be, e.g., $\varepsilon^{-1/2}$; then the above quantity can be made as small as we like by choosing ε large enough; this implies the bound

$$\mathbb{E}\left[\left(\left\langle\prod_{(hk)\in\Gamma}\psi_h\psi_k\right\rangle_0^{\psi}\right)^6\right]^{1/3}\leqslant e^{-K_{17}|\Gamma|}$$

where $K_{17}(\varepsilon)\to\infty$ as $\varepsilon\to\infty$.

Summarizing we have

$$(8)\leqslant\beta^{2|\Gamma|}\,\|F(\psi+\hat{h})\|_{\infty}^2\,e^{K_{16}|X|-(K_{17}-K_{15})|\Gamma|}$$

which gives Lemma 5 for $r=2$. The general case $r\geqslant 2$ is then easily obtained.

This completes the proof.

5. THE GROUND STATE OF THE ONE-DIMENSIONAL ISING MODEL WITH RANDOM MAGNETIC FIELD

In this section we will prove Theorem 3.

First we need the following easy lemma.

Lemma 6. Let $\mathscr{I}$ be an interval on $\mathbb{Z}$ (i.e., a connected set of nearest neighbor sites $[N,..., M]$), and let $\partial\mathscr{I}$ be its "boundary" $\{N-1, M+1\}$. Let $\sigma^{(\alpha,\beta)}$ be the ground state configuration with boundary conditions $\sigma_{N-1}=\alpha$, $\sigma_{M+1}=\beta$ $(\alpha,\beta=\pm 1)$ on $\partial\mathscr{I}$. Then

$$\mathbb{E}[\sigma_i^{(+,-)}]=\mathbb{E}[\sigma_i^{(-,+)}]=0 \tag{9}$$

$$\mathbb{E}[\sigma_i^{(+,+)}]\geqslant 0,\qquad \mathbb{E}[\sigma_i^{(-,-)}]\leqslant 0 \tag{10}$$

for each $i\in\mathscr{I}$.

Proof of Lemma 6. (9) follows immediately by symmetry. The inequalities (10) follow by considering $\mathbb{E}[\sigma_i^{(\alpha,\beta)}]$ as limit $\beta\to\infty$ of expectations at nonzero temperature, for which the following:

$$\mathbb{E}[\langle\sigma_i\rangle_{\beta,\Lambda}^{(+,+)}]\geqslant 0$$

$$\mathbb{E}[\langle\sigma_i\rangle_{\beta,\Lambda}^{(-,-)}]\leqslant 0$$

holds by the FKG inequalities. The proof is complete. ∎

Proof of Theorem 3. Let $\mathscr{B}=[-L,..., L]$ and let $\sigma(h)$ be the ground state configuration with $(+,+)$ boundary conditions on $\partial\mathscr{B}$. Then we want to prove

$$\mathbb{E}[\sigma_0(h)]\to 0\qquad\text{as}\quad L\to\infty$$

We define the following events:

$$\mathscr{E}_1 = \left\{ \exists l \mid -L \leqslant l < 0, h_l < -\frac{2J}{\varepsilon} \right\}$$

$$\mathscr{E}_2 = \left\{ \exists k \mid 0 < k \leqslant L, h_k < -\frac{2J}{\varepsilon} \right\}$$

If $\mathscr{E}_1$ (resp. $\mathscr{E}_2$) holds, then surely σ_l (resp. σ_k) $= -1$. Let $\chi_a = \chi_{\mathscr{E}_a}$, $\bar{\chi}_a = 1 - \chi_a$, $a = 1, 2$. Let also $p = \mathbb{P}[h_i < -2J/\varepsilon]$, $q = 1 - p$. Then clearly $\mathbb{E}[\bar{\chi}_a] = q^L \to 0$ as $L \to \infty$.

We use Lemma 6 and some trivial estimates to get

$$\begin{aligned} 0 \leqslant \mathbb{E}[\sigma_0] &= \mathbb{E}[\sigma_0 \chi_1 \chi_2] + \mathbb{E}[\sigma_0 \chi_1 \bar{\chi}_2] + \mathbb{E}[\sigma_0 \bar{\chi}_1 \chi_2] + \mathbb{E}[\sigma_0 \bar{\chi}_1 \bar{\chi}_2] \\ &\leqslant \mathbb{E}[\sigma_0 \chi_1 \chi_2] + 3q^L \end{aligned}$$

If we prove $\mathbb{E}[\sigma_0 \chi_1 \chi_2] \leqslant 0$ we are obviously done. To this aim, we write χ_1, χ_2 as follows:

$$\chi_1 = \sum_{-L \leqslant l < 0} \chi\left(h_l < -\frac{2J}{\varepsilon}; \forall x < l: h_x \geqslant -\frac{2J}{\varepsilon}\right) = \sum_{-L \leqslant l < 0} \chi_{1,l}$$

$$\chi_2 = \sum_{0 < k \leqslant L} \chi\left(h_k < -\frac{2J}{\varepsilon}; \forall x > k: h_x \geqslant -\frac{2J}{\varepsilon}\right) = \sum_{0 < k \leqslant L} \chi_{2,k}$$

Then we can write

$$\mathbb{E}[\sigma_0 \chi_1 \chi_2] = \sum_{\substack{-L \leqslant l < 0 \\ 0 < k \leqslant L}} \mathbb{E}[\sigma_0 \chi_{1,l} \chi_{2,k}]$$

and we want to prove

$$\mathbb{E}[\sigma_0 \chi_{1,l} \chi_{2,k}] \leqslant 0 \tag{11}$$

The characteristic functions $\chi_{1,l}$, $\chi_{2,k}$ impose $(-,-)$ boundary conditions on the subinterval $[l+1, k-1]$ containing 0. Therefore by Lemma 6 and by some elementary considerations based on the locality of the Ising Hamiltonian and the independence of the h_is we have (11).

The exponential decay of $\mathbb{E}[\sigma_i \sigma_j]$ can be easily proved by the following method.

Define the following event:

$$\hat{\mathscr{E}} = \left\{ \exists \hat{\mathscr{B}} \,\middle|\, |\hat{\mathscr{B}}| = n, \hat{\mathscr{B}} \text{ between } i \text{ and } j: \sum_{k \in \hat{\mathscr{B}}} h_k < -\frac{2J}{\varepsilon} \right\}$$

Let also $\hat{\chi} = \chi_{\hat{\mathscr{B}}}$, $\bar{\hat{\chi}} = 1 - \hat{\chi}$. It is easily seen that $\hat{\mathscr{B}}$ must contain at least a site k such that $\sigma_k = -1$. The size n of the interval will be chosen later. We then have

$$\mathbb{E}[\sigma_i \sigma_j] = \mathbb{E}[\sigma_i \hat{\chi} \sigma_j] + \mathbb{E}[\sigma_i \bar{\hat{\chi}} \sigma_j] \tag{12}$$

It is easily seen by a method similar to the one used to prove (11) that the first term in (12) is zero, so we are led to bound:

$$\begin{aligned} \mathbb{E}[\sigma_i \bar{\hat{\chi}} \sigma_j] &\leqslant \mathbb{E}[\bar{\hat{\chi}}] \\ &= \mathbb{P}\left[\forall \hat{\mathscr{B}}, |\hat{\mathscr{B}}| = n, \hat{\mathscr{B}} \text{ between } i \text{ and } j\text{: } \sum_{k \in \hat{\mathscr{B}}} h_k \geqslant -\frac{2J}{\varepsilon}\right] \end{aligned} \tag{13}$$

Choose $n = c/\varepsilon^2$ (we are led to this choice by the idea that the scaling law $\mathbb{E}[\sigma_0 \sigma_{sk}]_\varepsilon \approx \mathbb{E}[\sigma_0 \sigma_k]_{s^{1/2}\varepsilon}$ should hold). Then we bound (13) by considering only disjoint subintervals $\hat{\mathscr{B}}$ of length n between i and j (there are $|i-j|/n = \varepsilon^2 |i-j|/c$ such subintervals):

$$\begin{aligned} (13) &\leqslant \mathbb{P}\left[\forall \hat{\mathscr{B}} \text{ in a collection of disjoint subintervals, } |\hat{\mathscr{B}}| = \frac{c}{\varepsilon^2}, \right. \\ &\qquad \left. \hat{\mathscr{B}} \text{ between } i \text{ and } j\text{: } \sum_{k \in \hat{\mathscr{B}}} h_k \geqslant -\frac{2J}{\varepsilon}\right] \\ &= \prod_{\substack{\hat{\mathscr{B}} \text{disjoint}, \\ |\hat{\mathscr{B}}| = n, \hat{\mathscr{B}} \text{between } i \text{ and } j}} \mathbb{P}\left[\sum_{k \in \hat{\mathscr{B}}} h_k \geqslant -\frac{2J}{\varepsilon}\right] \leqslant \hat{q}^{\varepsilon^2 |i-j|/c} \end{aligned}$$

where $\hat{q}$ is independent of ε thanks to our choice of n. End of proof. ∎

6. CONCLUDING REMARKS

(a) A high-temperature cluster expansion for a class of random Ising models was developed by G. Gallavotti in Ref. 2.

(b) Stability estimates for a wider class of random spin systems have been obtained by Griffiths and Lebowitz[3]; see also Ref. 4.

(c) It is a pleasure to thank T. Spencer and A. Sokal for many helpful discussions and suggestions, and J. Fröhlich for having posed the problems and stimulated my interest. The author also acknowledges that J. Imbrie and J. Fröhlich have obtained similar results, by more powerful techniques.

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